A rigorous proof that heat flow is reversible if the temperature difference is infinitesimal

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Abstract

In thermodynamics, a reversible process is commonly defined as one in which both the system and its surroundings can be restored to their initial states. Heat transfer across an infinitesimal temperature difference is then presented as a standard example of a reversible process, although the connection between these two statements is not demonstrated explicitly.

This article provides a direct construction of that connection without using ideal constant-temperature reservoirs. A body undergoes successive heating and cooling contacts with a chain of N auxiliary bodies, each having the same constant heat capacity as the body itself. The contact dynamics are based on the elementary assumption that two such bodies, when thermally isolated from their surroundings and brought into contact with no work being performed, reach an equilibrium temperature equal to the average of their initial temperatures. All calculations are carried out in full, so that the reader need not reconstruct them and can concentrate on the physical reasoning.

It is proved that, as N tends to infinity, the main body approaches the prescribed higher temperature T_h at the end of the heating process and returns to its initial temperature T_c at the end of the cooling process. The maximum temperature difference across which heat is transferred tends to zero, and no finite amount of energy remains displaced among the bodies: the universe is restored in this limiting sense. Within the model considered here, this provides an explicit finite-heat-capacity demonstration of why heat transfer across an infinitesimal temperature difference is reversible.

Contents

1 How the problem has been addressed

Heat flow between bodies at an infinitesimal temperature difference is reversible: this idea underlies the reversibility of Carnot's engine, and therefore, in a sense, the second law and thermodynamics as a whole. Maxwell and other pioneers of thermodynamics recognized its importance. In [MAXWELL, pp. 149–150] it is stated:

> The peculiarity of Carnot's engine is, that whether it is receiving heat from the hot body, or giving it out to the cold body, the temperature of the substance in the engine differs extremely little from that of the body in thermal communication with it. [...] we may make the actual difference of temperature which causes the flow of heat to take place as small as we please. [...] an exceedingly small alteration of temperature will be sufficient to reverse the flow of heat, if the motion is slow enough. [...] by working the engine sufficiently slowly these [temperature] differences may be reduced within any limits we please to assign, so that for theoretical purposes we may regard Carnot's engine as strictly reversible.

However, Maxwell does not develop this observation into a proof. Later classic thermodynamics textbooks, such as [PLANCK] and [FERMI], do not seem to discuss the matter at all. In some modern textbooks the issue is occasionally mentioned, but in my view it has not been addressed satisfactorily. This article aims to examine it more thoroughly. Three instances will be briefly described.

a) In [FEYNMAN, Vol. I, Sec. 44.3] it is said that two bodies with an infinitesimal temperature difference can experience reversible heat flows. However, the point is argued rather vaguely, and a proof is lacking.

b) In [MORAN-SHAPIRO, p. 181] two bodies at different temperatures, able to communicate thermally, are considered. It is said that with a finite temperature difference between them the spontaneous heat transfer would be a source of irreversibility, but the importance of this irreversibility would diminish as the temperature difference approaches zero. Replacing a proof by the phrase "it might be expected" is sometimes acceptable. Here, however, it is not clear what guarantees that a process carried out through heat flows between bodies at arbitrarily close temperatures is reversible in this limit.

c) Similarly, in [STEANE, p. 24] two bodies in perfect thermal contact and at an infinitesimal temperature difference are considered. The author underlines that a change in circumstances by just more than δT will reverse the direction of the heat flow. I think that, although this conclusion is evident, it is not clear why the process must be reversible in the technical sense introduced a few lines earlier in the text: "A reversible process is one such that the system can be restored to its initial state without any net change in the rest of the universe".

In brief, in the treatments examined above, I have not found a clear explanation of the link between the “no net change in the universe” definition of a reversible process and the reversibility of heat flows between bodies with an infinitesimal temperature difference. Yet this link is important because it demonstrates that the concept of Carnot’s engine, reversible by definition, is meaningful. In these treatments, reversibility is simply taken for granted.

One treatment, however, does establish the link, at the price of a particularly strong idealization: that of ideal constant-temperature reservoirs. A solution within this framework was proposed by Chet Miller in an answer published on Physics Stack Exchange[1]. He considered a sequence of constant-temperature reservoirs whose temperatures differ by an arbitrarily small amount. A body is heated by placing it successively in contact with reservoirs at increasing temperatures and is then cooled by using them in the reverse order. All the intermediate reservoirs exchange equal and opposite amounts of heat during the two stages. Only the two reservoirs at the ends of the sequence fail to return exactly to their initial energies, and the residual amount of energy transferred between them tends to zero as the temperature spacing tends to zero. More precisely, the construction can be formulated by considering a finite chain of reservoirs and then taking the limit as their number tends to infinity.

2 Aim of the present article

The construction based on ideal thermal reservoirs provides a shorter proof, but achieves this brevity by assuming systems that maintain a constant temperature while exchanging energy. The present article develops a related but distinct thought experiment, inspired by Miller’s construction, in which ideal thermal reservoirs are dispensed with. Each auxiliary body has finite heat capacity and therefore undergoes a temperature change whenever it exchanges heat.

The article therefore addresses a more concrete question: can such a finite-heat-capacity construction produce the same reversible limit? The answer is affirmative. The greater mathematical complexity is the price of a more explicit and constructive physical description, in which the temperature and energy evolution of every body is followed. It is proved that, in the limit, the main body approaches the prescribed temperature $T_h > T_c$ and subsequently returns to its initial temperature T_c. In the same limit, every individual heat transfer takes place across a temperature difference that tends to zero, and no finite amount of energy remains displaced among the bodies. The article does not propose the shortest route to the result. Rather, it shows that the idealization of constant-temperature reservoirs is not necessary for obtaining this reversible limit.

Naturally, the limiting idealization of arbitrarily long chains remains and is treated mathematically by considering finite chains and taking the limit. Compared with the reservoir construction, however, the present model avoids the additional idealization of systems with infinite heat capacity. All the bodies

[1] `https://physics.stackexchange.com/a/251390`, accessed 13 August 2026.

involved obey the same elementary dynamics: two thermally isolated bodies with the same constant heat capacity, when brought into contact and allowed to reach equilibrium without performing work, attain the average of their initial temperatures. This does not make the reservoir-based proof any less rigorous, but it makes the construction presented here more concrete and, in this respect, more physically transparent: the evolution of each body and the redistribution of energy are explicitly followed and verified in full detail.

The finite-heat-capacity thought experiment is described in the following section.

3 Description of the thought experiment

Consider a body ξ at temperature T_c and let us suppose we want to heat it to a temperature $T_h > T_c$ and then to cool it to the initial temperature T_c, with a process that restores the initial state (nothing must be changed in the universe). Is it possible? Strictly speaking, no, but we can imagine a procedure that, as a limiting result, reaches the goal. Of course, we can put ξ in thermal contact with an identical body at temperature $2T_h - T_c$ and then with one at temperature $2T_c - T_h$ (provided that $T_c > T_h/2$). In this way we achieve the goal of bringing the temperature of ξ from T_c to T_h and then back to T_c, but at the cost of altering the universe: the two auxiliary bodies experience a change in temperature $2T_h - T_c \to T_h$ and $2T_c - T_h \to T_c$.

One may think of bypassing this issue by employing two thermal reservoirs instead of bodies with finite heat capacity. Their immense heat capacity would allow the body to be heated and then cooled, leaving the universe in its initial state. But this approach is flawed: during the process a finite quantity of energy has been transferred from one thermal reservoir to another, and even if we can minimize the temperature variations between the initial and final states (by choosing sufficiently large thermal capacities for the reservoirs), a finite amount of energy has unquestionably changed its position: the universe is *not* restored to its initial state. The fact that we can approximately restore the same energy density as at the beginning is irrelevant. This being the case, reversibility is not guaranteed. Indeed, a deeper analysis shows that the process just described with thermal reservoirs is not reversible: the total entropy of the universe increases by a finite amount (but it would be circular to discuss entropy without first having demonstrated that the heat exchange between bodies at very similar temperatures is reversible). This objection does not apply to Miller's limiting construction based on an increasingly long sequence of reservoirs, in which the residual energy transferred between the two end reservoirs tends to zero. Here, however, we shall pursue the different finite-heat-capacity construction announced in the preceding section.

As stated, there is a method to achieve the goal of performing a heating and cooling cycle for ξ while leaving the universe in its initial state: it is sufficient to use a long "thermal chain" of auxiliary bodies of finite heat capacity, with temperatures in the range from T_c to T_h. Every one of them therefore changes

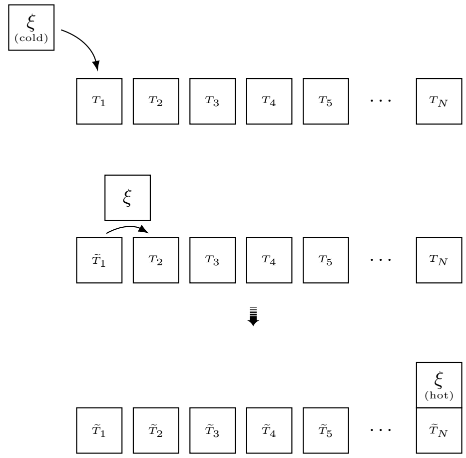

Figure 1: The heating process. T_n denotes the initial temperature of the n^{th} auxiliary body, so that T_1 is near the initial temperature of ξ. A tilde denotes the temperature reached by an auxiliary body after its contact with ξ. The dots stand for a large number of intermediate bodies.

its own temperature at each contact, and no constant-temperature reservoir is involved. In the limit of increasingly long thermal chains, the body ξ can be brought from T_c to T_h and back to T_c, without producing any net change in the universe. Fig. 1 provides a schematic representation of the thought experiment, which will be analysed in detail in the following sections. To simplify the calculations, I will further assume that all the auxiliary bodies have the same heat capacity as the body ξ.

4 The heating process

4.1 Proof that the equilibrium temperature when ξ is put in thermal contact with the n^{th} auxiliary body is (2)

Suppose we have N auxiliary bodies, each having the same constant heat capacity as ξ, with initial temperatures

$$T_c + \frac{T_h - T_c}{N} \qquad T_c + 2\frac{T_h - T_c}{N} \qquad \ldots \qquad T_c + (N-1)\frac{T_h - T_c}{N} \qquad T_h, \quad (1)$$

and imagine subsequently putting ξ (initially at temperature T_c) in thermal contact with these bodies (first with the body at temperature $T_c + (T_h - T_c)/N$, then with the body at temperature $T_c + 2(T_h - T_c)/N$, etc.). After equilibrium is reached at each step, ξ is separated from the auxiliary body before being placed in thermal contact with the next one. I claim that the equilibrium temperature when ξ is put in thermal contact with the n^{th} auxiliary body is given by

$$\frac{\left(2^{n}(N-n+1)-1\right)T_c+\left(2^{n}\,(n-1)+1\right)T_h}{2^{n}N} \qquad n=1,\ldots,N, \tag{2}$$

where N is the total number of bodies in the thermal chain, while n labels the body involved in the step under consideration, whose temperature is

$$T_c+\frac{n}{N}(T_h-T_c). \tag{3}$$

I shall prove this claim by induction.

1) The formula (2) holds for $n=1$ because it is easy to see that after thermal contact with the first body, the equilibrium temperature is

$$\frac{(2N-1)\,T_c+T_h}{2N}. \tag{4}$$

2) If the equilibrium temperature when ξ is put in thermal contact with the n^{th} auxiliary body is given by (2), then its equilibrium temperature when it is put in thermal contact with the $(n+1)^{\text{th}}$ auxiliary body, whose temperature is $T_c+(n+1)(T_h-T_c)/N$, is given by the average value:

$$\frac{\left(2^{n+1}(N-n)-1\right)T_c+\left(2^{n+1}n+1\right)T_h}{2^{n+1}N} \qquad n=1,\ldots,N-1, \tag{5}$$

but this is exactly what we find if we change n to $n+1$ in (2). This completes the proof.

4.2 Proof that the final temperature of the heating process approaches T_h as $N\to\infty$

Note that taking the limit $N\to\infty$ in (2) with n held fixed is not the relevant limit: using infinitely many auxiliary bodies but stopping at a fixed step leaves ξ arbitrarily close to its initial temperature, and the limit is simply T_c. The final temperature must therefore be obtained first, by setting $n=N$:

$$\frac{\left(2^{N}-1\right)T_c+\left(2^{N}\,(N-1)+1\right)T_h}{2^{N}N}. \tag{6}$$

This tends to T_h as $N\to\infty$, so the goal of bringing ξ to temperature T_h is achieved. Strictly speaking T_h is never attained by this process: what has just been proved is that ξ can be brought as close to T_h as desired by increasing the number of auxiliary bodies.

4.3 Proof that every temperature jump in the heating process tends to zero as $N\to\infty$

One may ask whether every temperature jump tends to zero as $N\to\infty$, that is, as the number of auxiliary bodies grows. It is intuitive, but I shall prove it formally here. In what follows, we shall consider only the temperature jumps of ξ. Since ξ and each auxiliary body have the same heat capacity, and no work is done during the thermal contacts, each auxiliary body undergoes a temperature change equal in magnitude and opposite in sign to that of ξ. Therefore, the same conclusion automatically applies to all the auxiliary bodies.

It is worth noting explicitly that the temperature difference between the two bodies at the moment of contact is exactly twice the temperature change undergone by each of them: if ξ is at temperature T and the auxiliary body at temperature T', the common equilibrium temperature $(T + T')/2$ differs from each of the two initial temperatures by $|T' - T|/2$. Consequently, every bound established below for the temperature jumps applies, up to a factor of two, to the temperature differences across which heat is actually transferred, and the two quantities tend to zero together.

Let us consider the temperature increase of ξ between the equilibrium reached after contact with the n^{th} auxiliary body and that reached after contact with the $(n+1)^{\text{th}}$ auxiliary body. For $n = 1, \ldots, N-1$, it is obtained by subtracting (2) from (5):

$$\Delta T_{n\to n+1} = \left(1 - \frac{1}{2^{n+1}}\right)\left(\frac{T_h - T_c}{N}\right). \tag{7}$$

The first temperature jump, from the initial state of ξ to the equilibrium reached after contact with the first auxiliary body, is obtained from (4)

$$\Delta T_1 = \frac{(2N-1)T_c + T_h}{2N} - T_c = \frac{T_h - T_c}{2N}. \tag{8}$$

This is precisely the value obtained from the preceding expression by setting $n = 0$. Therefore, by shifting the index, we can state the following general result: when ξ is placed in contact with the n^{th} body, it undergoes a temperature change

$$\Delta T_n = \left(1 - \frac{1}{2^n}\right)\left(\frac{T_h - T_c}{N}\right) \qquad n = 1, \ldots, N. \tag{9}$$

Since the factor

$$1 - \frac{1}{2^n} \tag{10}$$

is strictly increasing with n, the maximum temperature jump occurs during the last thermal contact, corresponding to $n = N$. Hence

$$\Delta T_{\text{max,heating}}(N) = \left(1 - \frac{1}{2^N}\right)\left(\frac{T_h - T_c}{N}\right). \tag{11}$$

It follows that

$$\lim_{N\to\infty} \Delta T_{\text{max,heating}}(N) = 0. \tag{12}$$

Every temperature jump in the heating process is less than or equal to the maximum one. Therefore, all the temperature jumps can be made simultaneously as small as desired by taking N sufficiently large. This completes the proof.

5 The cooling process

5.1 Proof that the equilibrium temperature when ξ is put in thermal contact with the $(N-i)^{\text{th}}$ auxiliary body is (13)

Let us now consider what happens when the thermal chain is traversed in the opposite direction, by placing ξ successively in contact with auxiliary bodies at progressively lower temperatures. At the end of the heating process, ξ is in thermal equilibrium with the N^{th} auxiliary body. The first contact of the cooling process is therefore with the $(N-1)^{\text{th}}$ body. I do not include a figure for this stage, since the arrangement is the one already shown in Fig. 1 on page 6, with only the order of the contacts reversed.

I claim that after thermal contact with the $(N-i)^{\text{th}}$ auxiliary body the equilibrium temperature is given by

$$\frac{\left(2^N\left(2^i i+1\right)-\frac{2^{2i+1}+1}{3}\right)T_c+\left(\frac{2^{2i+1}+1}{3}+2^N\left(2^i(N-i)-1\right)\right)T_h}{2^{N+i}N}, \tag{13}$$

where $i=1,\ldots,N-1$. Again, I shall prove this claim by induction.

1) Setting $n=N-1$ in (2), we find that after the heating process the $(N-1)^{\text{th}}$ auxiliary body remains at the equilibrium temperature it reached with ξ:

$$\frac{\left(2^{N+1}-2\right)T_c+\left(2^N\left(N-2\right)+2\right)T_h}{2^N N}, \tag{14}$$

and putting it in thermal contact with ξ (which, at the end of the heating process, is at the temperature given by (6)) we find an equilibrium temperature that can be rearranged as

$$\frac{3\left(2^N-1\right)T_c+\left(2^N\left(2N-3\right)+3\right)T_h}{2^{N+1}N}. \tag{15}$$

This is the same temperature we find by setting $i=1$ in (13), so the formula holds for the first thermal contact in the cooling process.

2) After the heating process, the $(N-i-1)^{\text{th}}$ auxiliary body takes the temperature given by (2) with $n=N-i-1$:

$$\frac{\left(2^{N-i-1}\left(i+2\right)-1\right)T_c+\left(2^{N-i-1}\left(N-i-2\right)+1\right)T_h}{2^{N-i-1}N}, \tag{16}$$

where $i=1,\ldots,N-2$. If the equilibrium temperature when ξ is put in thermal contact with the $(N-i)^{\text{th}}$ auxiliary body is given by (13), then its equilibrium temperature when it is put in thermal contact with the $(N-i-1)^{\text{th}}$ auxiliary body, whose temperature is (16), is given by their average

(the numerator has been split over two lines because of its length)

$$\frac{\begin{array}{c}\left(2^N\left(2^{i+1}\left(i+1\right)+1\right)-\frac{2^{2i+3}+1}{3}\right)T_c+\\ \left(\frac{2^{2i+3}+1}{3}+2^N\left(2^{i+1}(N-i-1)-1\right)\right)T_h\end{array}}{2^{N+i+1}N}, \tag{17}$$

but this is exactly what we find if we change $N-i$ to $N-i-1$ (i.e. i to $i+1$) in (13). This completes the proof.

5.2 Proof that the final temperature of the cooling process approaches T_c as $N \to \infty$

As before, taking the limit $N \to \infty$ in (13) with i held fixed is not the relevant limit: using infinitely many auxiliary bodies but stopping at a fixed step leaves ξ arbitrarily close to the temperature it had at the end of the heating process, and the limit is simply T_h. The final temperature must therefore be obtained first, by setting $i = N-1$:

$$\frac{\left(2^{2N-1}(3N-4)+3\cdot 2^N-1\right)T_c+\left(2^{2N+1}-3\cdot 2^N+1\right)T_h}{3\cdot 2^{2N-1}N}. \tag{18}$$

This tends to T_c as $N \to \infty$: the body ξ therefore returns to its initial temperature. Strictly speaking T_c is never attained exactly, but ξ can be brought as close to it as desired by increasing the number of auxiliary bodies.

5.3 Proof that every temperature jump in the cooling process tends to zero as $N \to \infty$

As in the heating process, it is sufficient to consider the temperature jumps of ξ, since during each thermal contact the auxiliary body undergoes a temperature change equal in magnitude and opposite in sign to that of ξ. At the end of the heating process, ξ has reached thermal equilibrium with the N^{th} auxiliary body. The cooling process therefore begins with the $(N-1)^{\text{th}}$ auxiliary body and ends with the first one. Notice that (13) also holds for $i=0$: in this case it reduces to (6), which gives the temperature of ξ at the end of the heating process. Since (17) is nothing but (13) with i replaced by $i+1$, it holds for $i = 0, \ldots, N-2$ as well. Consequently, for $N \geq 2$ and $i = 0, \ldots, N-2$, the temperature change undergone by ξ when it passes from the equilibrium reached after contact with the $(N-i)^{\text{th}}$ auxiliary body to that reached after contact with the $(N-i-1)^{\text{th}}$ auxiliary body is obtained by subtracting (13) from (17). We obtain the following negative temperature jumps:

$$\Delta T_{N-i \to N-i-1} = \left(1-\frac{1}{2^{i+1}}+\frac{1-2^{2i+2}}{3\cdot 2^{i+N+1}}\right)\left(\frac{T_c-T_h}{N}\right). \tag{19}$$

Let

$$A_{N,i} = 1-\frac{1}{2^{i+1}}+\frac{1-2^{2i+2}}{3\cdot 2^{i+N+1}}. \tag{20}$$

We have

$$i \geq 0 \quad \Rightarrow \quad 1 - 2^{2i+2} < 0 \quad \Rightarrow \quad \frac{1 - 2^{2i+2}}{3 \cdot 2^{i+N+1}} < 0 \quad \Rightarrow \quad A_{N,i} < 1 - \frac{1}{2^{i+1}}, \tag{21}$$

and therefore

$$A_{N,i} < 1. \tag{22}$$

Moreover,

$$\frac{1}{3 \cdot 2^{i+N+1}} > 0 \quad \Rightarrow \quad A_{N,i} > 1 - \frac{1}{2^{i+1}} - \frac{2^{2i+2}}{3 \cdot 2^{i+N+1}} = 1 - \left(\frac{1}{2^{i+1}} + \frac{2^{i+1-N}}{3} \right). \tag{23}$$

For $i = 0, \ldots, N-2$, we have

$$\left(\frac{1}{2^{i+1}} \leq \frac{1}{2} \quad \wedge \quad 2^{i+1-N} \leq \frac{1}{2} \right) \quad \Rightarrow \quad \frac{1}{2^{i+1}} + \frac{2^{i+1-N}}{3} \leq \frac{2}{3}, \tag{24}$$

and therefore $A_{N,i} > \frac{1}{3}$, which implies

$$A_{N,i} > 0. \tag{25}$$

From (22) and (25) we have

$$0 < A_{N,i} < 1 \tag{26}$$

for every cooling contact.

Since $T_h > T_c$, taking absolute values in (19) gives

$$0 < |\Delta T_{N-i \to N-i-1}| < \frac{T_h - T_c}{N}, \qquad i = 0, \ldots, N-2. \tag{27}$$

Let

$$\Delta T_{\max,\text{cooling}}(N) = \max_{0 \leq i \leq N-2} |\Delta T_{N-i \to N-i-1}|. \tag{28}$$

The preceding inequality gives

$$0 < \Delta T_{\max,\text{cooling}}(N) < \frac{T_h - T_c}{N}. \tag{29}$$

Since $T_h - T_c$ is finite,

$$\lim_{N \to \infty} \Delta T_{\max,\text{cooling}}(N) = 0. \tag{30}$$

Here too, the magnitude of every temperature jump is less than or equal to the maximum one: all the temperature jumps can therefore be made simultaneously as small as desired by taking N sufficiently large. This completes the proof.

6 Why all the energy is returned to its original position

6.1 Proof that the statement "all the energy is returned to its original position" is equivalent to condition (37)

Before proving anything, it is worth making clear what exactly needs to be proved. This is less trivial than it may appear, because conditions that look equivalent to the statement that all the energy has returned to its original position turn out to be weaker, as Appendix B.4 shows. The aim of this section is therefore to identify the condition that actually expresses that statement.

As mentioned in Section 3, in order to talk about a reversible process we must be sure that everything can be restored to the original situation, and in particular that all the energy transferred from one body to another has returned to the bodies in which it was initially located. To make this precise, let $\Delta E_j^{(N)}$ denote the difference between the final and the initial energy of body j, where j runs over ξ and all the N auxiliary bodies. Since the whole set of bodies is isolated from the rest of the universe, energy conservation gives

$$\sum_j \Delta E_j^{(N)} = 0. \tag{31}$$

Let

$$E_N^+ = \sum_{\Delta E_j^{(N)} > 0} \Delta E_j^{(N)} \tag{32}$$

be the total excess energy of the bodies whose final energy is greater than their initial energy, and let

$$E_N^- = -\sum_{\Delta E_j^{(N)} < 0} \Delta E_j^{(N)} \tag{33}$$

be the total energy missing from the remaining bodies. Energy conservation implies

$$E_N^+ = E_N^-. \tag{34}$$

Moreover,

$$\begin{aligned} \sum_j \left|\Delta E_j^{(N)}\right| &= \sum_{\Delta E_j^{(N)} > 0} \Delta E_j^{(N)} + \sum_{\Delta E_j^{(N)} < 0} \left(-\Delta E_j^{(N)}\right) \\ &= E_N^+ + E_N^- \\ &= 2E_N^+. \end{aligned} \tag{35}$$

Thus,

$$E_N^+ = E_N^- = \frac{1}{2} \sum_j \left|\Delta E_j^{(N)}\right|. \tag{36}$$

This quantity is precisely the amount of energy that remains displaced: it is the excess energy accumulated in the bodies that ended up with more energy than they started with, and equivalently the energy missing from those that ended up with less. The factor $\frac{1}{2}$ matters for the interpretation of the quantity, though not for the limit: each unit of displaced energy is counted twice in the sum, once as an excess and once as a deficit. It is therefore omitted from the limiting condition below, since multiplication by a non-zero constant does not affect whether a quantity tends to zero. Therefore, proving that

$$\lim_{N\to\infty} \sum_j \left|\Delta E_j^{(N)}\right| = 0 \tag{37}$$

is equivalent to proving that no finite amount of energy remains displaced at the end of the process.

6.2 Proof that (37) holds in our model

Since all the bodies have the same constant heat capacity C, and no work is done, the final energy variation of each body is equal to C times its final temperature variation.

Let us first consider the auxiliary bodies. For $i = 0, \ldots, N-1$, the final temperature of the $(N-i)^{\text{th}}$ auxiliary body can be obtained from (13), which was written above with the coefficients of T_c and T_h displayed separately, also reflecting the form in which it was originally conjectured (see Appendix A.2). For the present calculation it is more convenient to rewrite it in the following equivalent form:

$$T_{N-i}^{\text{final}} = T_c + \frac{(T_h - T_c)}{N}\left(N - i - \frac{1}{2^i} + \frac{2^{i+1-N} + 2^{-N-i}}{3}\right). \tag{38}$$

For $i = 0$, this expression gives the final temperature of the N^{th} auxiliary body, which is not contacted again during the cooling process. As already observed, in this case (13) reduces to (6).

The initial temperature of the $(N-i)^{\text{th}}$ auxiliary body is

$$T_{N-i}^{\text{initial}} = T_c + \frac{N-i}{N}(T_h - T_c). \tag{39}$$

Consequently, its final energy variation is

$$\Delta E_{N-i}^{(N)} = \frac{C(T_h - T_c)}{N}\left(-\frac{1}{2^i} + \frac{2^{i+1-N} + 2^{-N-i}}{3}\right), \qquad i = 0, \ldots, N-1. \tag{40}$$

Taking the absolute value and using the triangle inequality, we obtain

$$\left|\Delta E_{N-i}^{(N)}\right| \leq \frac{C(T_h - T_c)}{N}\left(\frac{1}{2^i} + \frac{2^{i+1-N} + 2^{-N-i}}{3}\right). \tag{41}$$

Therefore,

$$\sum_{i=0}^{N-1}\left|\Delta E_{N-i}^{(N)}\right| \leq \frac{C(T_h-T_c)}{N}\left[\sum_{i=0}^{N-1}\frac{1}{2^i}+\frac{1}{3}\sum_{i=0}^{N-1}\left(2^{i+1-N}+2^{-N-i}\right)\right]. \qquad (42)$$

The two sums give

$$\sum_{i=0}^{N-1}\frac{1}{2^i}=2-2^{1-N} \qquad \text{and} \qquad \sum_{i=0}^{N-1}\left(2^{i+1-N}+2^{-N-i}\right)=2-2^{1-2N}, \qquad (43)$$

so

$$\sum_{i=0}^{N-1}\left|\Delta E_{N-i}^{(N)}\right| \leq \frac{C(T_h-T_c)}{N}\left(\frac{8}{3}-2^{1-N}-\frac{2^{1-2N}}{3}\right). \qquad (44)$$

Since exponentials are positive,

$$\sum_{i=0}^{N-1}\left|\Delta E_{N-i}^{(N)}\right| < \frac{8C(T_h-T_c)}{3N}. \qquad (45)$$

It remains to consider ξ. At the end of the last cooling contact, ξ and the first auxiliary body are in thermal equilibrium and therefore have the same final temperature. Thus, by setting $i=N-1$ in (38), we find

$$T_\xi^{\text{final}}=T_c+\frac{(T_h-T_c)}{N}\left(\frac{4}{3}-2^{1-N}+\frac{2^{1-2N}}{3}\right). \qquad (46)$$

Since the initial temperature of ξ is T_c, its final energy variation is

$$\Delta E_\xi^{(N)}=\frac{C(T_h-T_c)}{N}\left(\frac{4}{3}-2^{1-N}+\frac{2^{1-2N}}{3}\right). \qquad (47)$$

For $N\geq 2$, we have

$$2^{1-N}=\left(\frac{1}{2}\right)^{N-1}\leq\frac{1}{2} \qquad \text{and} \qquad 0<\frac{2^{1-2N}}{3}=\frac{1}{3}\left(\frac{1}{2}\right)^{N}2^{1-N}<2^{1-N}. \qquad (48)$$

Consequently,

$$\frac{4}{3}-2^{1-N}+\frac{2^{1-2N}}{3}>\frac{4}{3}-2^{1-N} \qquad \Rightarrow \qquad \frac{4}{3}-2^{1-N}+\frac{2^{1-2N}}{3}\geq\frac{5}{6}>0, \qquad (49)$$

while

$$\frac{4}{3}-2^{1-N}+\frac{2^{1-2N}}{3}<\frac{4}{3}-2^{1-N}+2^{1-N} \qquad \Rightarrow \qquad \frac{4}{3}-2^{1-N}+\frac{2^{1-2N}}{3}<\frac{4}{3}. \qquad (50)$$

Therefore,

$$0<\frac{4}{3}-2^{1-N}+\frac{2^{1-2N}}{3}<\frac{4}{3}, \qquad (51)$$

and hence

$$\left|\Delta E_\xi^{(N)}\right| < \frac{4C(T_h - T_c)}{3N}. \tag{52}$$

Combining the bounds obtained for ξ and for all the auxiliary bodies, we find

$$\begin{aligned}\sum_j \left|\Delta E_j^{(N)}\right| &= \left|\Delta E_\xi^{(N)}\right| + \sum_{i=0}^{N-1} \left|\Delta E_{N-i}^{(N)}\right| \\ &< \frac{4C(T_h - T_c)}{3N} + \frac{8C(T_h - T_c)}{3N} \\ &= \frac{4C(T_h - T_c)}{N}.\end{aligned} \tag{53}$$

Since C and $T_h - T_c$ are finite,

$$\lim_{N\to\infty} \sum_j \left|\Delta E_j^{(N)}\right| = 0. \tag{54}$$

Thus condition (37) holds: no finite amount of energy remains displaced among the bodies at the end of the process. Therefore, in the limit $N \to \infty$, all the energy has returned to its original position.

7 Conclusions

The starting point of this article is the elementary assumption that two bodies with the same constant heat capacity, thermally isolated from their surroundings and brought into contact without performing work, reach an equilibrium temperature equal to the average of their initial temperatures. From this alone one can construct a sequence of finite thermal chains such that, as the number N of auxiliary bodies tends to infinity:

- the maximum temperature jump occurring during both the heating and the cooling processes tends to zero,
- the body ξ approaches the prescribed temperature T_h and subsequently returns to its initial temperature T_c,
- no finite amount of energy remains displaced among the bodies.

Thus, within the model considered here, every heat transfer takes place across an arbitrarily small temperature difference, and yet the system and its environment return to their initial states in the limit $N \to \infty$. This is precisely the connection announced at the outset: such a process satisfies the "no net change in the universe" definition of a reversible process. The construction also shows that this reversible limit can be obtained without ideal constant-temperature reservoirs, using only bodies with finite heat capacity whose complete thermal and energetic evolution is followed explicitly. For this reason, despite its greater mathematical complexity, I find the present construction particularly transparent from a physical point of view.

A Where the formulae proved by mathematical induction come from

As often happens with proofs by mathematical induction, the reader may wonder where the formulae being proved come from, since they may look as if they had "fallen from the sky". It is not necessary to answer this question to make the proof "more rigorous", and a completely legitimate answer could be "it was found by a stroke of luck". That, however, is not the answer here, and it may be of interest to show the reasoning that led me to conjecture (2) and (13).

A.1 How formula (2) was found

Putting ξ in thermal contact with the first body, I found the equilibrium temperature

$$\frac{(2N-1)\,T_c + T_h}{2N} \qquad (n=1) \tag{4}$$

Putting ξ then in thermal contact with the second body, and going on with the next bodies, I obtained these equilibrium temperatures

$$\frac{(4N-5)\,T_c + 5T_h}{4N} \qquad (n=2) \tag{55}$$

$$\frac{(8N-17)\,T_c + 17T_h}{8N} \qquad (n=3) \tag{56}$$

$$\frac{(16N-49)\,T_c + 49T_h}{16N} \qquad (n=4) \tag{57}$$

$$\frac{(32N-129)\,T_c + 129T_h}{32N} \qquad (n=5) \tag{58}$$

$$\frac{(64N-321)\,T_c + 321T_h}{64N} \qquad (n=6) \tag{59}$$

It is clear what is happening with the denominators. By consulting *The On-Line Encyclopedia of Integer Sequences*, I was able to conjecture that the sequence $1, 5, 17, 49, 129, 321, \ldots$ appearing in the numerators is given by $\{(n-1)\cdot 2^n + 1\}$. This led me to conjecture (2) for the equilibrium temperature when ξ is put in thermal contact with the n^{th} auxiliary body. The conjecture is then rigorously proved in the article.

A.2 How formula (13) was found

Assuming (2) to be correct, I found that if, after the whole heating process, ξ is put in thermal contact with the $(N-1)^{\text{th}}$ auxiliary body, the equilibrium temperature is given by (15). I reproduce it here without factoring out the common factor 3, for reasons that will become clear below:

$$\frac{\left(2^N\cdot 3 - 3\right) T_c + \left(2^N\,(2N-3) + 3\right) T_h}{2^{N+1}N} \qquad (i=1) \tag{15}$$

Putting ξ then in thermal contact with the $(N-2)^{\text{th}}$ auxiliary body, which after the heating process has reached the temperature given by (2) with $n = N-2$,

$$\frac{(3 \cdot 2^{N-2} - 1)T_c + \left(2^{N-2}\,(N-3) + 1\right) T_h}{2^{N-2} N}, \tag{60}$$

I found an equilibrium temperature

$$\frac{\left(2^N \cdot 9 - 11\right) T_c + \left(2^N\,(4N-9) + 11\right) T_h}{2^{N+2} N} \qquad (i = 2) \tag{61}$$

Going on in this way, after thermal contact with the $(N-3)^{\text{th}}$ and the $(N-4)^{\text{th}}$ auxiliary bodies, I found

$$\frac{\left(2^N \cdot 25 - 43\right) T_c + \left(2^N\,(8N-25) + 43\right) T_h}{2^{N+3} N} \qquad (i = 3) \tag{62}$$

and

$$\frac{\left(2^N \cdot 65 - 171\right) T_c + \left(2^N\,(16N-65) + 171\right) T_h}{2^{N+4} N} \qquad (i = 4) \tag{63}$$

By consulting *The On-Line Encyclopedia of Integer Sequences* I was able to conjecture that $3, 9, 25, 65, \ldots$ is the sequence $\{i \cdot 2^i + 1\}$, while $3, 11, 43, 171, \ldots$ is $\left\{(2^{2i+1} + 1)/3\right\}$. This led me to conjecture that after thermal contact with the $(N-i)^{\text{th}}$ auxiliary body the equilibrium temperature is given by (13). Again, the conjecture is rigorously proved in the article.

B History of this article

B.1 The origin of the article

The initial physical inspiration for this article came from an answer by Chet Miller on Physics Stack Exchange. Miller described how a body could be heated through a sequence of constant-temperature reservoirs separated by arbitrarily small temperature differences and then returned to its initial temperature by contacting the reservoirs in the reverse order. He observed that only the first and last reservoirs would fail to return exactly to their initial energies and that this residual effect would become insignificant in the limiting construction. I was particularly interested in whether the same physical idea could be realized without ideal constant-temperature reservoirs, using instead auxiliary bodies with finite heat capacity whose temperatures would actually change during every contact. The ideas developed in this article first took shape while I was on holiday in 2021, and an initial version of the resulting finite-heat-capacity construction was published on Physics Stack Exchange on 3 October 2021.

That answer already contained the central thought experiment, the heating and subsequent cooling of a body ξ by means of a long thermal chain of auxiliary bodies, together with both general formulae (2) and (13), the limit giving the final temperature of ξ at the end of the heating process, and a limiting argument, based on the parameter $x = n/N$ (and $x = (N - i)/N$ in the cooling process), that appeared to show that every auxiliary body returns to its initial temperature.

Three points still required work. The first was a proof of the general formulae, which had been conjectured from the first terms of numerical sequences with the aid of *The On-Line Encyclopedia of Integer Sequences*. I openly acknowledged at the time that these heuristic steps were not rigorous ("this step should be fixed"). The second was the limiting argument based on a parameter x giving the position along the chain, which required revision for the reasons explained below. The third was the complete absence of any argument concerning the *total* energy displaced: a gap I did not notice for a long time.

The present article grew out of the subsequent effort to turn that initial physical idea into a complete and rigorous argument.

B.2 The Brazilian reader

On 24 August 2023, I was contacted by a Brazilian reader, who wrote: "I found your response about reversible heat transfer on Physics Exchange very interesting." When I pointed out that the answer was only a preliminary draft, since some steps in the derivation were still missing, the reader kindly replied: "Don't worry if you can't analytically derive the formula for the temperature of the nth iteration. Consider it as an Ansatz."

Encouraged by this appreciation, I returned to the problem and, after some thought, I realized that the general formulae could be proved by mathematical induction. Eight days later, on 1 September 2023, I obtained rigorous proofs of both formulae. However, I did not publish the resulting answer on Physics Stack Exchange until 10 June 2024. It is still there in that form: I have not updated it (having stopped contributing to the platform at the beginning of February 2026).

More than two years later, in the summer of 2026, I asked ChatGPT to take a look at the work and tell me what it thought. The resulting discussion dispelled my belief that the argument had already been made completely rigorous and led me to re-examine it and recognize two points that still required correction.

B.3 The first issue identified by ChatGPT

A first issue identified by ChatGPT concerned the use of the parameter x in the original proof that every temperature jump becomes infinitesimal. In previous versions, x was kept fixed while taking the limit $N \to \infty$. However, in a thermal chain containing N auxiliary bodies, the index n must be an integer and x can assume only the discrete values

$$\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N}{N}. \tag{64}$$

Consequently, for a generic fixed value of x, the substitution $n = xN$ is not defined for every integer value of N. For example, $x = 0.1$ corresponds to $n = 10$

when $N = 100$ and to $n = 11$ when $N = 110$, but no auxiliary body corresponds exactly to $x = 0.1$ when $N = 101$. For $x \in \mathbb{Q}$, the original argument could have been made rigorous by restricting N to a suitable subsequence depending on x, but this would have introduced unnecessary complications. ChatGPT suggested replacing this argument with a more direct one: for each value of N, the largest temperature jump in the whole chain is considered. Since this maximum tends to zero as $N \to \infty$, every temperature jump is bounded by a quantity that tends to zero. In the case of the cooling process, ChatGPT proposed a mathematically elementary but particularly elegant and ingenious argument (see Section 5.3).

B.4 The second issue identified by ChatGPT

B.4.1 Why the original energy arguments were not sufficient

A second issue identified by ChatGPT concerned the claim that, at the end of the process, all the energy has returned to its original position. In the 2024 version I calculated the total heat absorbed and released by ξ and showed that its net energy variation tends to zero. As for the auxiliary bodies, I dismissed the question on the grounds that each of them undergoes at most two vanishing heat flows. ChatGPT pointed out that neither argument is sufficient. A vanishing net variation for ξ says nothing about how the energy is distributed among the auxiliary bodies (in principle ξ could remove a finite amount of energy from one part of the chain and deposit it in another, and still return to its own initial energy). Nor does the fact that each auxiliary body undergoes only vanishing heat flows say anything about their sum, since the number of bodies grows without bound as the limit is approached: a growing number of vanishing deviations may still add up to a finite amount. The principle was already at hand: in the very next sentence of that version I had applied it to ξ, for which the number of heat flows grows without bound, but not to the chain, whose number of bodies also grows without bound.

To see more precisely what the old argument did establish, let $\Delta E_\xi^{(N)}$ denote the final energy variation of ξ, and let $\Delta E_n^{(N)}$ denote that of the n^{th} auxiliary body. Unlike in Section 6.1, where the same energy balance is written with a single index running over all the bodies, here ξ is kept outside the sum, so that n retains its usual meaning of n^{th} auxiliary body. Since the complete set of bodies is isolated from the rest of the universe, energy conservation gives

$$\Delta E_\xi^{(N)} + \sum_{n=1}^{N} \Delta E_n^{(N)} = 0. \tag{65}$$

Consequently,

$$\Delta E_\xi^{(N)} \longrightarrow 0 \qquad \Longleftrightarrow \qquad \sum_{n=1}^{N} \Delta E_n^{(N)} \longrightarrow 0. \tag{66}$$

Thus, the condition proved for ξ and the vanishing of the algebraic sum of the energy variations of the auxiliary bodies are two equivalent expressions of the same energy balance. But this controls only the algebraic sum. Positive and negative variations may cancel, even when a finite amount of energy remains displaced among the auxiliary bodies. It therefore does not prove that the energy has returned to the bodies in which it was initially located.

B.4.2 Why the maximum variation is not sufficient either

ChatGPT further showed that the condition

$$\max_{1\le n\le N} \left|\Delta E_n^{(N)}\right| \longrightarrow 0 \tag{67}$$

would not be sufficient either. This condition guarantees that no individual auxiliary body remains appreciably altered, but it does not exclude the accumulation of a finite amount of displaced energy over a number of bodies that increases with N.

To make this distinction unmistakable, ChatGPT constructed the following counterexample. Restricting attention to even values of N, let $E_0 > 0$ be a fixed constant independent of N, and suppose that

$$\Delta E_n^{(N)} = \begin{cases} \dfrac{E_0}{N}, & 1 \le n \le N/2, \\ -\dfrac{E_0}{N}, & N/2 < n \le N. \end{cases} \tag{68}$$

Summing (68), we obtain

$$\sum_{n=1}^{N} \Delta E_n^{(N)} = 0. \tag{69}$$

Together with (65) this implies

$$\Delta E_\xi^{(N)} = 0. \tag{70}$$

Thus ξ returns exactly to its initial energy for every value of N considered. Moreover $|\Delta E_n^{(N)}| = \frac{E_0}{N}$, and therefore (67) holds: the magnitude of the energy variation of every individual auxiliary body tends to zero.

Nevertheless, the first half of the chain has acquired the finite amount of energy

$$\frac{N}{2}\frac{E_0}{N} = \frac{E_0}{2}, \tag{71}$$

which is precisely the amount lost by the second half. A finite amount of energy has therefore been transferred from one part of the chain to another. Indeed,

$$\sum_{n=1}^{N} \left|\Delta E_n^{(N)}\right| = N\frac{E_0}{N} = E_0, \tag{72}$$

so the condition (37) is not satisfied.

This simple counterexample showed that even the combination of (70) and condition (67) is not sufficient to ensure that all the energy has returned to its original position. Through the argument presented in Section 6.1, ChatGPT showed that condition (37) precisely expresses the physical claim that no finite amount of energy remains displaced among the bodies. Finally, by means of another elementary but elegant argument (see Section 6.2), ChatGPT proved that condition (37) is indeed satisfied in the model considered in this article.

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