I wonder if what is done here is totally correct.

6.2 INTERFERENCE OF TWO WAVES

Imagine that we have two linearly polarized plane waves of the same wavelength, given by

$$\mathbf{E}_1(\mathbf{r}, t) = \mathbf{E}_{01} \cos(\mathbf{k}_1 \cdot \mathbf{r} - \omega t + \varepsilon_1)$$
$$\mathbf{E}_2(\mathbf{r}, t) = \mathbf{E}_{02} \cos(\mathbf{k}_2 \cdot \mathbf{r} - \omega t + \varepsilon_2)$$

which overlap at point P as in Fig. 6-1. Here, of course, $\mathbf{k}_1$, $\mathbf{k}_2$, ω, ε_1 and ε_2 are all constant. These waves may arise, for example, from two, very distant, point sources. The resultant field is simply

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

But since at optical frequencies the fields oscillate at in excess of 10^{14} Hz the irradiance becomes the quantity of practical concern, it being directly measurable. Neglecting a constant factor, we can write the irradiance as just the time average of the total field:

$$I = \langle \mathbf{E}^2 \rangle$$

where $\mathbf{E}^2 \equiv \mathbf{E} \cdot \mathbf{E}$. Accordingly,

$$\mathbf{E}^2 = (\mathbf{E}_1 + \mathbf{E}_2) \cdot (\mathbf{E}_1 + \mathbf{E}_2) = \mathbf{E}_1^2 + \mathbf{E}_2^2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2$$

Taking the average we obtain

$$I = I_1 + I_2 + I_{12}$$

where $I_1 = \langle \mathbf{E}_1^2 \rangle$, $I_2 = \langle \mathbf{E}_2^2 \rangle$ and $I_{12} = 2\langle \mathbf{E}_1 \cdot \mathbf{E}_2 \rangle$, the last being known as the *interference term*. It is by virtue of this last term that I differs from a simple sum of the irradiances of the component waves, i.e. $I_1 + I_2$.

The time average of some function $f(t)$ is given by

$$\langle f(t) \rangle = \frac{1}{T} \int_t^{t+T} f(t')\, dt'$$

Here the detection interval T is much greater than the period of oscillation of the waves, τ. Thus if we carried out the indicated calculation for the above plane waves (see Problem 2.13), the interference term would become

$$I_{12} = 2\langle \mathbf{E}_1 \cdot \mathbf{E}_2 \rangle = \mathbf{E}_{01} \cdot \mathbf{E}_{02} \cos \delta$$

where the *phase difference* δ is given by

$$\delta = (\mathbf{k}_1 \cdot \mathbf{r}) - (\mathbf{k}_2 \cdot \mathbf{r}) + \varepsilon_1 - \varepsilon_2$$

All of this means that as one moves from point to point in space, $\mathbf{r}$ varies, as does δ, and therefore I_{12} and I both vary as well.

I try to explain my doubts. We define intensity as the power transferred per unit area perpendicular to the direction of propagation of the energy. If we have a plane wave this doesn't give problems, but here we have two waves and it is not clear which is the plane. So I think we should define intensity as a vector $\mathbf{S}$ such that $\mathbf{S} \cdot d\mathbf{a}dt = dE$, where dE is the energy that cross $d\mathbf{a}$ during $t \to t + dt$. Maxwell theory tell us that a such vector exists, and its value is $\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}$ and that if we have a plane wave $\mathbf{S} = c\epsilon_0 E^2 \hat{\mathbf{n}}$ where $\hat{\mathbf{n}} = \hat{\mathbf{E}} \times \hat{\mathbf{B}}$ (I use hat for versors and I wrote instantaneous values, I have not yet averaged over rapidly oscillating waves). But I can't understand how the book treat the case of two waves.

Let's try to work the problem: in $\mathbf{r}$ at time t the fields are $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ and $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$ (I use subscript to denote the wave). Calculating $\mathbf{S}$ we get

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \frac{1}{\mu_0}(\mathbf{E}_1 \times \mathbf{B}_2 + \mathbf{E}_2 \times \mathbf{B}_1)$$

But if $\hat{\mathbf{n}}$ is the direction of an electromagnetic wave $\mathbf{B} = \frac{\hat{\mathbf{n}}\times\mathbf{E}}{c}$ so

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \epsilon_0 c[\mathbf{E}_1 \times (\hat{\mathbf{n}}_2 \times \mathbf{E}_2) + \mathbf{E}_2 \times (\hat{\mathbf{n}}_1 \times \mathbf{E}_1)]$$

This can be rearranged with the triple vector product identity, but the result it doesn't look nicer or useful, so I leave it in that way. If the two waves are harmonic, $\mathbf{E}_1 = \mathbf{E}_{01}\cos(\mathbf{k}_1 \cdot \mathbf{r} - \omega t + \epsilon_1)$ and $\mathbf{E}_2 = \mathbf{E}_{02}\cos(\mathbf{k}_2 \cdot \mathbf{r} - \omega t + \epsilon_2)$, and we get

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{X}\cos(\mathbf{k}_1 \cdot \mathbf{r} - \omega t + \epsilon_1)\cos(\mathbf{k}_2 \cdot \mathbf{r} - \omega t + \epsilon_2)$$

where I used the *constant* vector

$$\mathbf{X} = \epsilon_0 c[\mathbf{E}_{01} \times (\hat{\mathbf{n}}_2 \times \mathbf{E}_{02}) + \mathbf{E}_{02} \times (\hat{\mathbf{n}}_1 \times \mathbf{E}_{01})]$$

So

$$< \mathbf{S} >=< \mathbf{S}_1 > + < \mathbf{S}_2 > +\mathbf{X} < \cos(\mathbf{k}_1 \cdot \mathbf{r} - \omega t + \epsilon_1)\cos(\mathbf{k}_2 \cdot \mathbf{r} - \omega t + \epsilon_2) >$$

Exploiting $\cos(a-b) = \cos a \cos b + \sin a \sin b$ we have that $< \cos\left[(\mathbf{k_1} \cdot \mathbf{r} + \epsilon_1) - \omega t\right] \cos\left[(\mathbf{k_2} \cdot \mathbf{r} + \epsilon_2) - \omega t\right] >$ is

$$< [\cos(\mathbf{k_1} \cdot \mathbf{r} + \epsilon_1)\cos(\omega t) + \sin(\mathbf{k_1} \cdot \mathbf{r} + \epsilon_1)\sin(\omega t)] \cdot [\cos(\mathbf{k_2} \cdot \mathbf{r} + \epsilon_2)\cos(\omega t) + \sin(\mathbf{k_2} \cdot \mathbf{r} + \epsilon_2)\sin(\omega t)] > \tag{1}$$

But if the wave oscillates rapidly we can exploit $< \cos^2(\omega t) >=< \sin^2(\omega t) >= \frac{1}{2}$ and $< \cos(\omega t)\sin(\omega t) >= 0$, so doing products and observing again $\cos(a-b) = \cos a \cos b + \sin a \sin b$ we conclude that

$$< \mathbf{S} >=< \mathbf{S}_1 > + < \mathbf{S}_2 > +\mathbf{X}\cos(\mathbf{k_1} \cdot \mathbf{r} + \epsilon_1 - \mathbf{k_2} \cdot \mathbf{r} - \epsilon_2)$$

This recalls the book equation, but it is not the same because I use vectors and because the I_{12} term is very different (I used the $\mathbf{X}$ vector, ugly, but isn't it necessary?). So I have two question:

- Is it correct in this two waves problem to treat intensity as a scalar, as book does? (the books has a general approach, it doesn't make hypothesis like $\hat{\mathbf{n}}_1 \approx \hat{\mathbf{n}}_2$, on the contrary a figure explicitly show that waves moves in very different directions)
- Why my equation is so different from the book's one, where did I wrong? Is it possible simply find intensity averaging $|\mathbf{E}|^2$ as we do when we have a single wave?

Edit

The key is that Poynting says that the only important thing to find intensity vector in a given point at a given time are total $\mathbf{E}$ and $\mathbf{B}$, but if we have many waves overlapping the total $\mathbf{E}$ and $\mathbf{B}$ are *in some way* independent. I mean that they can have any relative direction and th ratio of magnitude is not c. How can we so conoclude that, in a generic overlapping of two waves, the intensity in a give place $\mathbf{r}$ and time t is proportional to the square of local electric field? This can be demonstrated for one wave (by exploiting $\mathbf{E} \perp \mathbf{B}$ and $E = Bc$) but not in a general overlapping of waves, as it is assumed in the book proof.

By the way, if the two waves moves approximately in the same direction ($\hat{\mathbf{n}}_1 \approx \hat{\mathbf{n}}_2 = \hat{\mathbf{n}}$), the book reasoning gives approximately the correct result (using triple vector product identity with my vector $\mathbf{X}$ and the fact that $\mathbf{E}_{01}, \mathbf{E}_{02} \perp \hat{\mathbf{n}}$), but this hypothesis is not assumed, the approach is general, so it looks an error. I wrong?